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LETTER TO THE EDITOR

Prolongation structure and linear eigenvalue equations for Einstein–Maxwell fields

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Abstract. The Einstein–Maxwell equations for stationary axisymmetric exterior fields are shown to be the integrability conditions of a set of linear eigenvalue equations for pseudopotentials. It turns out that the prolongation structure (in the spirit of Wahlquist and Estabrook) of the Einstein–Maxwell equations contains the SU(2, 1) Lie algebra. A new mapping of known solutions to other solutions has been found.

1. The Wahlquist–Estabrook method

The Wahlquist–Estabrook (1975) method can be summarised as follows. Let us start with a given system of N first-order differential equations

$$f^{\Omega}(x^1, x^2, U^B, U^B_{,1}, U^B_{,2}) = 0, \qquad \Omega = 1, \dots, N, \tag{1}$$

for some unknown functions $U^A(x^1, x^2)$, $A = 1, \dots, n$, of two independent variables x^1 and x^2. The so-called pseudopotentials q^k are defined by

$$\omega^k = -\mathrm{d}q^k + F^k(x^1, x^2, U^A; q^l)\mathrm{d}x^1 + G^k(x^1, x^2, U^A; q^l)\mathrm{d}x^2 = 0, \qquad \mathrm{d}\omega^k = 0, \tag{2}$$

provided that the differential equations (1) are satisfied. The functions F^k and G^k in (2) depend on x^1 and x^2, the variables U^A, and the pseudopotentials q^k. The occurrence of q^k itself as an argument of F^k and G^k gives rise to a generalisation of the ordinary notion of potential.

Suppose the differential equations (1) are written in terms of a closed set σ^α of differential forms. The closed set $(\sigma^\alpha, \omega^k)$ is called a prolongation of σ^α. The quantities (x^1, x^2, U^A) are said to be the primitive variables, and the prolongation procedure requires pseudopotentials q^k to be added.

The condition $\mathrm{d}\omega^k = 0$ in (2) leads to partial differential equations for F^k and G^k. In some cases the integration with respect to the primitive variables can be carried out explicitly and the problem reduces to equations for functions $X^k_a(q^l)$, $a = 1, \dots, r$, which depend on q^k alone. These remaining equations are called the prolongation structure. Appropriate additional assumptions restrict the prolongation structure to become a Lie algebra with commutators

$$[X_a, X_b] = \left(X^k_a \frac{\partial}{\partial q^k} X^l_b - X^k_b \frac{\partial}{\partial q^k} X^l_a\right) \frac{\partial}{\partial q^l} = C^c_{ab} X_c. \tag{3}$$

For the final determination of F^k and G^k one simply has to find a representation of that Lie algebra. The pseudopotentials q^k satisfy the equations

$$dq^k = F^k dx^1 + G^k dx^2 \tag{4}$$

with now determined functions F^k and G^k.

In the cases investigated so far, there are transformations

$$U'^A = U'^A(x^1, x^2, U^B, q^k) \tag{5}$$

which map known solutions (U^A) to more general solutions (U'^A) of the differential equations under consideration. The Bäcklund transformations of the Korteweg–de Vries equation were considered by Wahlquist and Estabrook (1975) as an example. Following these lines, Harrison (1978) derived transformations of the type (5) for the Ernst equation of the stationary axisymmetric vacuum problem in Einstein's theory. These transformations are contained in the more general treatment given by Belinsky and Zakharov (1978, 1979) and Neugebauer (1979, 1980).

The present paper is devoted to an application of the Wahlquist–Estabrook method to the Einstein–Maxwell equations.

2. The Einstein–Maxwell field equations

In terms of the Ernst potential $\mathscr{E}$ and the scalar electromagnetic potential Φ, the Einstein–Maxwell equations for stationary axisymmetric exterior fields reduce to the system

$$(\mathrm{Re}\,\mathscr{E} + \Phi\bar{\Phi}) W^{-1}[(W\mathscr{E}_{,Z})_{,\bar{Z}} + (W\mathscr{E}_{,\bar{Z}})_{,Z}] = \mathscr{E}_{,Z}(\mathscr{E}_{,\bar{Z}} + 2\bar{\Phi}\Phi_{,\bar{Z}}) + \mathscr{E}_{,\bar{Z}}(\mathscr{E}_{,Z} + 2\bar{\Phi}\Phi_{,Z}), \tag{6a}$$

$$(\mathrm{Re}\,\mathscr{E} + \Phi\bar{\Phi}) W^{-1}[(W\Phi_{,Z})_{,\bar{Z}} + (W\Phi_{,\bar{Z}})_{,Z}] = \Phi_{,Z}(\mathscr{E}_{,\bar{Z}} + 2\bar{\Phi}\Phi_{,\bar{Z}}) + \Phi_{,\bar{Z}}(\mathscr{E}_{,Z} + 2\bar{\Phi}\Phi_{,Z}), \tag{6b}$$

of second-order differential equations. The real function W obeys the Laplace equation

$$W_{,Z\bar{Z}} = 0. \tag{6c}$$

The complex conjugate coordinates $x^1 = Z$ and $x^2 = \bar{Z}$ are the non-ignorable coordinates of stationary axisymmetric space–times.

In terms of the quantities $A_1, \ldots, E_1$, defined by

$$\begin{aligned} A_1 &:= \frac{\mathscr{E}_{,Z} + 2\bar{\Phi}\Phi_{,Z}}{2(\mathrm{Re}\,\mathscr{E} + \bar{\Phi}\Phi)}, \qquad B_1 := \frac{\bar{\mathscr{E}}_{,Z} + 2\Phi\bar{\Phi}_{,Z}}{2(\mathrm{Re}\,\mathscr{E} + \bar{\Phi}\Phi)}, \qquad C_1 := \frac{W_{,Z}}{W}, \\ D_1 &:= \frac{\mathrm{i}\bar{\Phi}_{,Z}}{(\mathrm{Re}\,\mathscr{E} + \bar{\Phi}\Phi)^{1/2}}, \qquad E_1 := \frac{\mathrm{i}\Phi_{,Z}}{(\mathrm{Re}\,\mathscr{E} + \bar{\Phi}\Phi)^{1/2}}, \end{aligned} \tag{7}$$

and $A_2, \ldots, E_2$, defined by the corresponding relations with $\bar{Z}$ in place of Z, the field equations (6) can be written in the first-order form (Herlt 1980)

$$\begin{aligned}
A_{1,\bar{Z}} &= A_1A_2 - A_1B_2 - \tfrac{1}{2}C_1A_2 - \tfrac{1}{2}C_2A_1 - E_1D_2,\\
A_{2,Z} &= A_1A_2 - A_2B_1 - \tfrac{1}{2}C_1A_2 - \tfrac{1}{2}C_2A_1 - E_2D_1,\\
B_{1,\bar{Z}} &= B_1B_2 - B_1A_2 - \tfrac{1}{2}C_1B_2 - \tfrac{1}{2}C_2B_1 - E_2D_1,\\
B_{2,Z} &= B_1B_2 - B_2A_1 - \tfrac{1}{2}C_1B_2 - \tfrac{1}{2}C_2B_1 - E_1D_2,\\
C_{1,\bar{Z}} &= -C_1C_2,\\
C_{2,Z} &= -C_1C_2,\\
D_{1,\bar{Z}} &= B_1D_2 + \tfrac{1}{2}B_2D_1 - \tfrac{1}{2}C_1D_2 - \tfrac{1}{2}C_2D_1 - \tfrac{1}{2}A_2D_1,\\
D_{2,Z} &= B_2D_1 + \tfrac{1}{2}B_1D_2 - \tfrac{1}{2}C_1D_2 - \tfrac{1}{2}C_2D_1 - \tfrac{1}{2}A_1D_2,\\
E_{1,\bar{Z}} &= A_1E_2 + \tfrac{1}{2}A_2E_1 - \tfrac{1}{2}C_1E_2 - \tfrac{1}{2}C_2E_1 - \tfrac{1}{2}B_2E_1,\\
E_{2,Z} &= A_2E_1 + \tfrac{1}{2}A_1E_2 - \tfrac{1}{2}C_1E_2 - \tfrac{1}{2}C_2E_1 - \tfrac{1}{2}B_1E_2.
\end{aligned} \tag{8}$$

3. The prolongation structure

The condition $d\omega^k = 0$ can be integrated with respect to the primitive variables $(Z, \bar{Z}, A_1, \ldots, E_1, A_2, \ldots, E_2)$; one obtains the solution

$$\begin{aligned}
F^k &= X_1^kA_1 + X_2^kB_1 + X_3^kE_1 + X_4^kD_1 + \lambda(X_5^kA_1 + X_6^kB_1 + X_7^kE_1 + X_8^kD_1),\\
G^k &= X_1^kA_2 + X_2^kB_2 + X_3^kE_2 + X_4^kD_2 + \lambda^{-1}(X_5^kA_2 + X_6^kB_2 + X_7^kE_2 + X_8^kD_2),
\end{aligned} \tag{9}$$

where λ is determined by

$$\lambda_{,Z} = \tfrac{1}{2}\lambda(\lambda^2-1)C_1, \qquad \lambda_{,\bar{Z}} = \tfrac{1}{2}\lambda^{-1}(\lambda^2-1)C_2. \tag{10}$$

The functions X_a^k, $a = 1, \ldots, 8$, which depend only on q^l, have to satisfy the following prolongation structure relations (note that $X_a := X_a^k\partial/\partial q^k$):

$$\begin{aligned}
&[X_1, X_2] + [X_5, X_6] = X_2 - X_1, \qquad [X_3, X_4] + [X_7, X_8] = X_2 - X_1,\\
&[X_1, X_3] + [X_5, X_7] = \tfrac{1}{2}X_3, \qquad [X_2, X_3] + [X_6, X_7] = \tfrac{1}{2}X_3,\\
&[X_1, X_4] + [X_5, X_8] = \tfrac{1}{2}X_4, \qquad [X_2, X_4] + [X_6, X_8] = \tfrac{1}{2}X_4,\\
&[X_1, X_5] = -[X_2, X_5] = -X_5, \qquad [X_1, X_6] = -[X_2, X_6] = -X_6,\\
&[X_1, X_7] = -[X_2, X_7] = -\tfrac{1}{2}X_7, \qquad [X_1, X_8] = -[X_2, X_8] = \tfrac{1}{2}X_8,\\
&[X_4, X_7] = X_5, \qquad [X_3, X_5] = -X_7, \qquad [X_4, X_6] = -X_8,\\
&[X_4, X_5] = [X_3, X_6] = [X_3, X_7] = [X_4, X_8] = 0.
\end{aligned} \tag{11}$$

Obviously, not all the commutators $[X_a, X_b]$ are completely determined. However, if we put $X_1 = -X_2$ and introduce $Y := [X_3, X_4]$ as a new generator, we obtain expressions for $[Y, X_a]$, and the additional information $[X_5, X_7] = 0$, $[X_6, X_8] = 0$, from the Jacobi identities. Inspection of table 1, which gives the full list of commutators, shows us that we have arrived at the SU(2, 1) algebra with the generators $Y, X_2, \ldots, X_8$.

Table 1. The list of commutators, e.g. $[Y, X_5] = X_5$.

	X_2	X_3	X_4	X_5	X_6	X_7	X_8
Y	0	$-2X_2$	$2X_4$	X_5	$-X_6$	$-X_7$	X_8
X_2		$-\frac{1}{2}X_3$	$\frac{1}{2}X_4$	X_5	$-X_6$	$\frac{1}{2}X_7$	$-\frac{1}{2}X_8$
X_3			Y	$-X_7$	0	0	X_6
X_4				0	$-X_8$	X_5	0
X_5					$2X_2$	0	X_4
X_6						X_3	0
X_7							$2X_2-Y$

This result implies that we can use the (nonlinear) representation $(q^1 \equiv \alpha, q^2 \equiv \beta)$

$$\begin{aligned} &-X_1^k = X_2^k = (\alpha, \tfrac{1}{2}\beta), \qquad X_3^k = (\beta, 0), \qquad X_4^k = (0, -\alpha), \\ &X_5^k = (-\alpha^2, -\alpha\beta), \qquad X_6^k = (1, 0), \qquad X_7^k = (\alpha\beta, \beta^2), \qquad X_8^k = (0, -1), \end{aligned} \tag{12}$$

which provides us with a solution of (11). From equations (4), (9) and (12) we obtain the simultaneous system of equations

$$\begin{aligned} \alpha_{,z} &= -\alpha A_1 + \alpha B_1 + \beta E_1 + \lambda(-\alpha^2 A_1 + B_1 + \alpha\beta E_1), \\ \alpha_{,\bar{z}} &= -\alpha A_2 + \alpha B_2 + \beta E_2 + \lambda^{-1}(-\alpha^2 A_2 + B_2 + \alpha\beta E_2), \\ \beta_{,z} &= -\tfrac{1}{2}\beta A_1 + \tfrac{1}{2}\beta B_1 - \alpha D_1 + \lambda(-\alpha\beta A_1 + \beta^2 E_1 - D_1), \\ \beta_{,\bar{z}} &= -\tfrac{1}{2}\beta A_2 + \tfrac{1}{2}\beta B_2 - \alpha D_2 + \lambda^{-1}(-\alpha\beta A_2 + \beta^2 E_2 - D_2), \end{aligned} \tag{13}$$

for the pseudopotentials α and β, with λ as in (10). In the vacuum case ($D_1 = D_2 = E_1 = E_2 = 0$), these equations decouple and α satisfies a total Riccati equation (see Neugebauer 1979).

4. The linear eigenvalue equations

The equations (13) can be linearised by putting

$$\alpha = \psi/\chi, \qquad \beta = \sigma/\chi. \tag{14}$$

It is easily verified that the linear relations

$$\begin{aligned} \psi_{,z} &= B_1(\psi + \lambda\chi) + E_1\sigma, \\ \chi_{,z} &= A_1(\chi + \lambda\psi) - \lambda E_1\sigma, \\ \sigma_{,z} &= \tfrac{1}{2}(A_1 + B_1)\sigma - D_1(\psi + \lambda\chi), \\ \psi_{,\bar{z}} &= B_2(\psi + \lambda^{-1}\chi) + E_2\sigma, \\ \chi_{,\bar{z}} &= A_2(\chi + \lambda^{-1}\psi) - \lambda^{-1}E_2\sigma, \\ \sigma_{,\bar{z}} &= \tfrac{1}{2}(A_2 + B_2)\sigma - D_2(\psi + \lambda^{-1}\chi), \end{aligned} \tag{15}$$

for the pseudopotentials ψ, χ, σ imply (13). The system (8) of the Einstein–Maxwell equations follows from the integrability conditions of (15). (Note that $A_1, \ldots, E_2$ do not contain the eigenvalue hidden in λ, so that in the integrability conditions the expressions with different powers of λ must vanish separately.) The equation (8) can also be deduced from the associated linear eigenvalue problem

$$\begin{aligned}
\tilde{\psi}_{,z} &= A_1(\tilde{\psi}+\lambda\tilde{\chi})+D_1\tilde{\sigma},\\
\tilde{\chi}_{,z} &= B_1(\tilde{\chi}+\lambda\tilde{\psi})-\lambda D_1\tilde{\sigma},\\
\tilde{\sigma}_{,z} &= \tfrac{1}{2}(A_1+B_1)\tilde{\sigma}-E_1(\tilde{\psi}+\lambda\tilde{\chi}),\\
\tilde{\psi}_{,\bar{z}} &= A_2(\tilde{\psi}+\lambda^{-1}\tilde{\chi})+D_2\tilde{\sigma},\\
\tilde{\chi}_{,\bar{z}} &= B_2(\tilde{\chi}+\lambda^{-1}\tilde{\psi})-\lambda^{-1}D_2\tilde{\sigma},\\
\tilde{\sigma}_{,\bar{z}} &= \tfrac{1}{2}(A_2+B_2)\tilde{\sigma}-E_2(\tilde{\psi}+\lambda^{-1}\tilde{\chi}),
\end{aligned} \tag{16}$$

for the pseudopotentials $\tilde{\psi}$, $\tilde{\chi}$, $\tilde{\sigma}$, which are related to ψ, χ, σ by

$$\tilde{\psi}(\lambda)=\overline{\psi(1/\bar{\lambda})}, \qquad \tilde{\chi}(\lambda)=\overline{\chi(1/\bar{\lambda})}, \qquad \tilde{\sigma}(\lambda)=-\overline{\sigma(1/\bar{\lambda})}. \tag{17}$$

For $\lambda=1$, the equations (15) and (16) can be integrated to give (up to trivial constants)

$$\mathscr{E}=\chi|_{\lambda=1}, \qquad \Phi=\frac{\mathrm{i}}{\sqrt{2}}\frac{\tilde{\sigma}}{(\psi+\chi)^{1/2}}\bigg|_{\lambda=1}, \tag{18}$$

for the Ernst and electromagnetic potentials.

5. Mapping of solutions

It turns out that the Einstein–Maxwell equations (8) remain invariant under the transformation

$$\begin{aligned}
&A_1'=\lambda\Big(-\frac{\psi}{\chi}A_1+\frac{\sigma}{\chi}E_1\Big), \qquad B_1'=\lambda\Big(-\frac{\tilde{\psi}}{\tilde{\chi}}B_1+\frac{\tilde{\sigma}}{\tilde{\chi}}D_1\Big), \qquad C_1'=\lambda^2C_1,\\
&D_1'=\lambda\Big(\frac{\chi}{\tilde{\chi}}\Big)^{1/2}\Big(\frac{\psi}{\tilde{\chi}}D_1+\frac{\sigma}{\tilde{\chi}}B_1\Big), \qquad E_1'=\lambda\Big(\frac{\tilde{\chi}}{\chi}\Big)^{1/2}\Big(\frac{\tilde{\psi}}{\chi}E_1+\frac{\tilde{\sigma}}{\chi}A_1\Big),\\
&A_2'=\frac{1}{\lambda}\Big(-\frac{\psi}{\chi}A_2+\frac{\sigma}{\chi}E_2\Big), \qquad B_2'=\frac{1}{\lambda}\Big(-\frac{\tilde{\psi}}{\tilde{\chi}}B_2+\frac{\tilde{\sigma}}{\tilde{\chi}}D_2\Big), \qquad C_2'=\frac{1}{\lambda^2}C_2,\\
&D_2'=\frac{1}{\lambda}\Big(\frac{\chi}{\tilde{\chi}}\Big)^{1/2}\Big(\frac{\psi}{\tilde{\chi}}D_2+\frac{\sigma}{\tilde{\chi}}B_2\Big), \qquad E_2'=\frac{1}{\lambda}\Big(\frac{\tilde{\chi}}{\chi}\Big)^{1/2}\Big(\frac{\tilde{\psi}}{\chi}E_2+\frac{\tilde{\sigma}}{\chi}A_2\Big).
\end{aligned} \tag{19}$$

The occurring pseudopotentials are solutions of (15) and (16) obeying the condition

$$\tilde{\psi}\psi+\tilde{\sigma}\sigma-\tilde{\chi}\chi=0. \tag{20}$$

The mapping (19) of solutions of the Einstein–Maxwell equations is a generalisation of the SU(2, 1) invariance transformation (see Kramer *et al* 1972, Kinnersley 1973).

Finally, we mention that there is still another kind of transformations preserving the equations (8):

$$\begin{aligned}
&A_1'=\alpha^{-1}A_1+\tfrac{1}{2}(1-\alpha^{-1})C_1, \qquad B_1'=B_1+\tfrac{1}{2}(1-\alpha)C_1, \qquad C_1'=C_1,\\
&A_2'=\alpha A_2+\tfrac{1}{2}(1-\alpha)C_2, \qquad B_2'=\alpha^{-1}B_2+\tfrac{1}{2}(1-\alpha^{-1})C_2, \qquad C_2'=C_2,\\
&E_1'=\alpha^{-1/2}E_1, \qquad D_1'=\alpha^{1/2}D_1, \qquad E_2'=\alpha^{1/2}E_2, \qquad D_2'=\alpha^{-1/2}D_2,
\end{aligned} \tag{21}$$

where

$$\begin{aligned} \alpha_{,Z} &= (\alpha^2-\alpha)(-B_1+\tfrac{1}{2}C_1)+(\alpha-1)(-A_1+\tfrac{1}{2}C_1), \\ \alpha_{,\bar{Z}} &= (\alpha^2-\alpha)(-A_2+\tfrac{1}{2}C_2)+(\alpha-1)(-B_2+\tfrac{1}{2}C_2). \end{aligned} \tag{22}$$

This transformation is equivalent to that given by Omote *et al* (1980). It corresponds to the freedom in choosing arbitrary linear combinations of the two Killing vectors in the stationary axisymmetric space–time.

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